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A GOODNESS-OF-FIT TEST FOR A CLASS OF NONHOMOGENEOUS POISSON PR--ETC(U)
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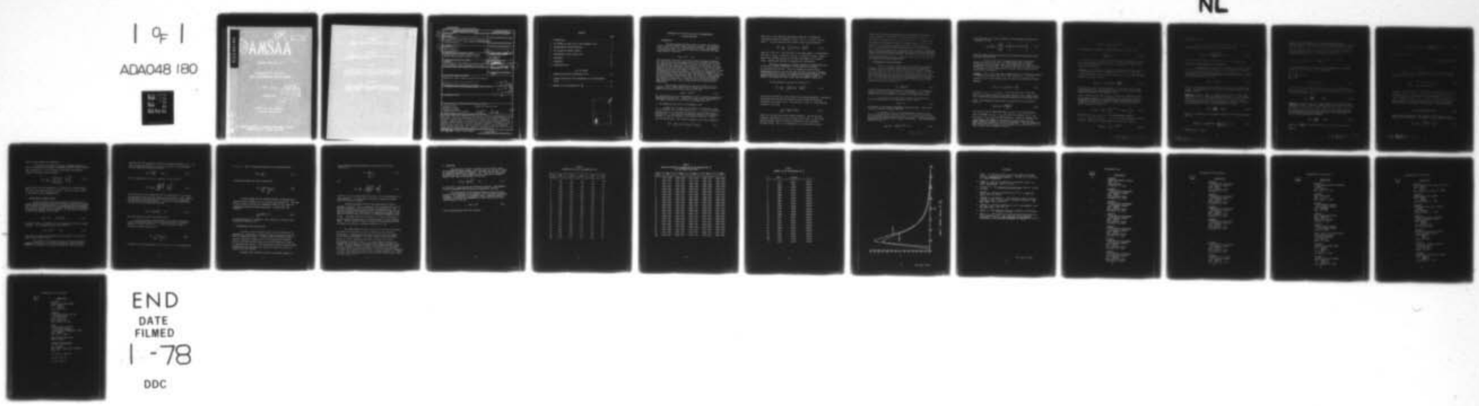

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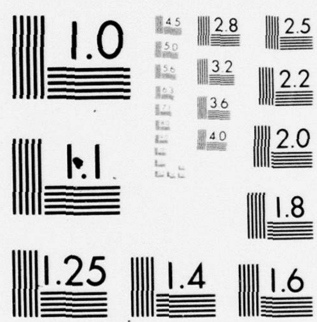
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AMSAA

TECHNICAL REPORT NO. 210

A GOODNESS-OF-FIT TEST FOR A
[C]LASS OF NONHOMOGENEOUS POISSON PROCESSES

DAVID L. CLARK

DECEMBER 1977

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

14

1. REPORT NUMBER
AMSAA-TR-210

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

6

4. TITLE (and Subtitle)
A Goodness-of-Fit Test for a Class of Nonhomogeneous Poisson Processes.

5. TYPE OF REPORT & PERIOD COVERED

6. PERFORMING ORG. REPORT NUMBER

10

7. AUTHOR(s)
David L./Clark

8. CONTRACT OR GRANT NUMBER(s)

9 Technical rept.

9. PERFORMING ORGANIZATION NAME AND ADDRESS
US Army Materiel Systems Analysis Activity
Aberdeen Proving Ground, Maryland 21005

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
DA Project No.
1R765706M541

11. CONTROLLING OFFICE NAME AND ADDRESS
Commander
US Army Materiel Development and Readiness Command
5001 Eisenhower Avenue, Alexandria, VA 22333

11

12. REPORT DATE
Dec[redacted]977

13. NUMBER OF PAGES
25

12 23p.

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; unlimited distribution.

16 1R765706M541

DDC
JAN 6 1978
F

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Goodness-of-fit
Reliability Growth
Parametric Exponent
Nonhomogeneous Poisson Process
Cramér-von Mises Statistic

403 910

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper discusses a goodness-of-fit test for a class of nonhomogeneous Poisson processes. This class of processes has been used to describe the phenomena of reliability growth and of the occurrence of failures in a complex, repairable system. Improved tables of percentiles of the small sample distribution of the Cramér-von Mises statistic in which a parametric exponent is estimated are provided for use in the test. The test can also be used to test the hypothesis that a random sample comes from a cumulative distribution function which can be expressed as an unknown positive power of a specified cdf.

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UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

CONTENTS

		Page
1.	INTRODUCTION	1
2.	THE CRAMÉR-von MISES TEST IN THE PARAMETRIC CASE	1
3.	NONHOMOGENEOUS POISSON PROCESSES	3
4.	THE RELIABILITY GROWTH PROCESS	9
5.	DISTRIBUTION OF THE STATISTIC C_m^2	11
6.	CONCLUSION	13
7.	REFERENCES	19
8.	DISTRIBUTION LIST	21

LIST OF TABLES

1.	PERCENTILES FOR THE DISTRIBUTION OF C_m^2	14
2.	INTERVAL ESTIMATES OF THE PERCENTILES OF THE DISTRIBUTION OF C_m^2	15
3.	MOMENTS OF THE DISTRIBUTION OF C_m^2	16

A GOODNESS-OF-FIT TEST FOR A CLASS OF NONHOMOGENEOUS POISSON PROCESSES

1. INTRODUCTION

The problem addressed here is that of testing the goodness-of-fit of certain integer valued stochastic processes. The hypothesis to be tested is that a set of waiting times are from a nonhomogeneous Poisson process which is a member of the class of processes with mean value function of the form

$$M(t) = \lambda t^{\beta} \quad t>0. \tag{1.1}$$

The occurrences of failures in military systems undergoing refinement in a test-fix-test-fix development process and in complex systems which are repaired upon failure are known to follow such processes. Crow (1) has discussed this application and shown how to use the Cramér-von Mises statistic to test the goodness-of-fit hypothesis. A table of the small sample distribution of the Cramér-von Mises statistic for the case in which an exponentially-appearing parameter is estimated also appeared in (1). Some percentiles of that table are inaccurate because of sample sizes used in the simulations which generated them. The objectives of the present work are to provide a more accurate table and to verify convergence to the asymptotic distribution. In addition various proofs were omitted from the previous work. Some of those proofs are included here for completeness.

The procedures presented here can also be used to test the hypothesis that a random sample from a continuous distribution has a cumulative distribution function $F(X;\theta)$ of the form

$$F(X;\theta) = (R(X))^{\theta} \tag{1.2}$$

for some positive value of the parameter θ and for a specified cumulative distribution function $R(X)$. Tables of the small sample distribution of the Cramér-von Mises test are provided to implement the test.

2. THE CRAMÉR-VON MISES TEST IN THE PARAMETRIC CASE

Cramér (2), von Mises (3), and Smirnov (4) developed a test of the hypothesis that a random sample $X_1, X_2, \ldots, X_N$ from a continuous distribution $G(X)$ is drawn from the completely specified distribution function $F(X)$. If the cumulative distribution function $F(X)$ contains a parameter θ, then the hypothesis to be tested is that $G(X) = F(X;\theta_o)$ for some specified θ_o. The statistic employed in the test is given by

$$W_n^2 = n \int_{-\infty}^{\infty} [F_n(X)-F(X;\theta_o)]^2 \, dF(X;\theta_o), \tag{2.1}$$

where $F_n(X)$ is the empirical distribution function. The empirical distribution function is defined as $F_n(X) = k/n$ if k of the X_i are less than X. It can be shown that the statistic above can be written as

$$W_n^2 = \frac{1}{12n} + \sum_{j=1}^{n} \left[F(X_j';\theta_o) - \frac{2j-1}{2n} \right]^2 \qquad (2.2)$$

where X_j' is the j-th order statistic of the random sample. The hypothesis that $G(X) = F(X;\theta_o)$ is rejected if W_n^2 is extraordinarily large. Smirnov derived the limiting distribution for W_n^2 as the sample size n becomes large. Anderson and Darling (5) tabulated this distribution.

Following a suggestion of Cramér, Darling (6) extended this test to the case in which the parameter θ is estimated by a statistic $\hat{\theta}_n$ calculated from the data. This test assumes that there exists a nondegenerate interval I on the real axis such that for every θ contained in the interior of I, $F(X;\theta)$ is a cumulative distribution function. The hypothesis tested is that the cdf from which the sample is drawn is a member of the parametric family $F(X;\theta)$ for some unknown value θ_o in I.

For the parametric case the test statistic is

$$C_n^2 = \frac{1}{12n} + \sum_{j=1}^{n} \left[F(X_j';\hat{\theta}_n) - \frac{2j-1}{2n} \right]^2, \qquad (2.3)$$

in which the estimate $\hat{\theta}_n$ is substituted for the unknown parameter. Darling (6) investigated the limiting distribution of C_n^2 and found that there are essentially two distinct cases. The first case is that of a superefficient estimator $\hat{\theta}$, such that

$$\lim_{n\to\infty} nE\left\{(\hat{\theta}_n-\theta)^2\right\}=0, \qquad (2.4)$$

where θ is the true value of the unknown parameter. In this case the asymptotic distribution of C_n^2 is the same as that of W_n^2. In the second and more general case the limiting distribution of C_n^2 is different from that of W_n^2. Thus it is inappropriate to use the tabled critical

values for W_n^2 to test the goodness-of-fit hypothesis in this case. If certain regularity conditions are satisfied and $\sqrt{n}\ (\hat{\theta}_n - \theta)$ is asympotically normal with mean zero and finite nonzero variance, then the limiting distribution of C_n^2 is the same as that of $\int_0^1 Y^2(t)\ dt$ where Y(t) is a Gaussian stochastic process with zero mean and a covariance kernel which depends upon the unknown true distribution G(X). Thus the C_n^2 test is not distribution free; however, in important special cases the distribution depends only upon the form of the family $f(X,\theta)$ and not upon the true value θ_0. Fortunately, for the case of an exponential parameter which is of concern in this report, the test has this property of being parameter free.

3. NONHOMOGENOUS POISSON PROCESSES

Parzen (7) defines the concept of a nonhomogeneous Poisson process. This is an integer valued process which has independent increments and unit jumps and which consists of the random variables N(t), the number of events occurring in the interval [0,t]. It follows from this definition that for any t>0, the random variable N(t) has the Poisson distribution with an expected value given by some function M(t). The mean value function M(t) is nondecreasing and is usually assumed to be continuous and differentiable. The derivation of the mean value function is denoted by

$$\nu(t) = \frac{d}{dt} M(t) \tag{3.1}$$

and is called the intensity function of the process. For a small increment of time h the quantity $\nu(t)$ h is approximately equal to the probability of the occurrence of an event in the interval (t,t+h).

In the special case for which the mean value function is directly proportional to the observation time t, that is,

$$M(t) = \lambda t \tag{3.2}$$

the process is the ordinary homogeneous Poisson process. Such a process has a constant intensity function.

In the general case of the nonhomogeneous Poisson process use of the characteristic function along with the property of independent increments reveals that for any interval (a,b) the number of events occurring in the interval has a Poisson distribution. The characteristic function of N(t) is

$$\phi_{N(t)}(u) = \exp[M(t)\{e^{iu} - 1\}] \quad . \tag{3.3}$$

Since the numbers of events occurring in nonoverlapping intervals are independent we have

$$\phi^{(u)}_{N(b)-N(a)} = \frac{\phi^{(u)}_{N(b)}}{\phi^{(u)}_{N(a)}} = \exp\left\{[M(b)-M(a)][e^{iu}-1]\right\} . \tag{3.4}$$

Thus the random variable $N(b)-N(a)$ has the Poisson distribution with expected value $M(b)-M(a)$.

A pair of well known theorems describe the relationship between the order statistics of a random sample from a uniform distribution and the waiting times to the occurrence of events in a homogeneous Poisson process. The proof of the first is given by Parzen. The statements of both theorems and the proofs are presented here to clarify this relationship.

Theorem 1. Let $\{N(t),\ t\geq 0\}$ be a Poisson process with constant intensity λ. Under the condition that $N(T)=n$, the times $t_1, t_2, \ldots t_n$ in the interval $(0,T]$ at which events occur have the joint probability density function

$$f(t_1, t_2, \cdots, t_n \mid N(T)=n) = \frac{n!}{T^n} \tag{3.5}$$

in which $0 < t_1 < t_2 < \cdots < t_n \leq T$. Note that this joint conditional density is the same as that of the order statistics of a random sample size n drawn from a uniform distribution defined on the interval $(0,T]$.

Proof: The probability that exactly n events occur in the interval $(0,T]$ is

$$P[N(T)=n] = \frac{(\lambda T)^n e^{-\lambda T}}{n!} . \tag{3.6}$$

Let $\{t_i\}$ be a set of times such that $0 < t_1 < t_2 < \cdots < t_n < T$ and choose a set of increments $\{h_1, h_2, \cdots, h_n\}$ such that the intervals $[t_1, t_1 + h_1]$, $[t_2, t_2+h_2], \cdots [t_n, t_n+h_n]$ are nonoverlapping. Since the increments are nonoverlapping the number of occurrences in each interval and the number occuring elsewhere in $(0,T]$ are independent. Thus the probability that exactly one event occurs in each interval $[t_i, t_i+h_i]$ for $i=1,2\ldots,n$ and no events occur elsewhere in $(0,T]$ is

$$\exp[-\lambda(T-\sum_{i=1}^{n} h_i)] \prod_{j=1}^{n} \lambda h_j e^{-\lambda h_j} \tag{3.7}$$

The conditional probability of this event given that $N(t) = n$ is

$$\frac{n!}{T^n} \prod_{i=1}^{n} h_i \ . \tag{3.8}$$

By allowing the h_i to decrease to infinitesimal increments, it follows that the conditional density of $t_1, t_2, \ldots, t_n$ given that $N(t)=N$ is as stated in the theorem.

Theorem 2. Let $\{N(t), t \geq 0\}$ be a Poisson process with constant intensity λ. Let t_n be the time of occurrence of the nth event. Under the condition that $t_n=T$, the n-1 waiting times $t_1, t_2, \ldots, t_{n-1}$ have the joint probability density function

$$f(t_1, t_2, \ldots, t_{n-1} \mid t_n = T) = \frac{(n-1)!}{T^{n-1}} \ . \tag{3.9}$$

Note that this point conditional density is the same as that of the order statistics of a random sample of size n-1 drawn from a uniform distribution defined on the interval (0,T].

Proof: Define $t_o=0$ and let $s_i=t_i-t_{i-1}$ be the i-th interarrival time of the Poisson process. It is known that the interarrival times are independently identically distributed random variables with the exponential distribution with mean $1/\lambda$. Thus the joint density of the first n interarrival times is

$$f(s_1, s_2, \ldots, s_n) = \prod_{i=1}^{n} \lambda e^{-\lambda s_i} = \lambda^n \exp[-\lambda \sum_{i=1}^{n} s_i] \tag{3.10}$$

for which $0 < s_1, s_2, \ldots, s_{n-1}, < \infty$. The transformation mapping $(t_1, t_2, \ldots, t_{n-1})$ into $(s_1, s_2, \ldots, {}_{n-1})$ has a Jacobian identically equal to unity. Therefore the joint density of the waiting times is

$$g(t_1, t_2, \ldots, t_n) = \lambda^n e^{-\lambda t_n} \tag{3.11}$$

with $0<t_1<t_2,\ldots<t_n$.

Since t_n is the convolution of n random variables from the same exponential distribution, it has the gamma density

$$g_n(t_n) = \frac{\lambda e^{-\lambda t_n}(\lambda t_n)^{n-1}}{(n-1)!} \quad t>0 \quad . \tag{3.12}$$

It follows that the conditional density of $t_1,\ldots,t_{n-1}$ given that $t_n=T$ is as in the statement of the theorem.

A nonhomogeneous process with a continuous mean value function M(t) can be transformed into a homogeneous Poisson process. Because M(t) is continuous and nondecreasing its inverse function $M^{-1}(X)$ can be defined for all $X>0$ as the minimum value of t such that $M(t)\geq X$. Define the stochastic process $\{K(X),X\geq 0\}$

$$K(X) = N(M^{-1}(X)) \quad . \tag{3.13}$$

This is a Poisson process with an intensity function identically equal to one. This transformation will be used in the proof of the two subsequent theorems.

Theorem 3. Let $\{N(t),t\geq 0\}$ be a Poisson process with continuous mean value function M(t). Under the condition that N(T) = n, the n waiting times $t_1,t_2,\ldots,t_n$ in the interval (0,T] at which events occur are random variables having the same distribution as the order statistics of a random sample of size n from the probability density

$$h(X) = \frac{\nu(X)}{M(T)} \qquad 0\leq X\leq T \tag{3.14}$$

where $\nu(X) = dM(X)/dx$ is the intensity function of the Poisson process. That is,

$$g(t_1,t_2,\ldots,t_n \mid N(T)=n) = \frac{n!}{[M(T)]^n} \prod_{i=1}^{n} \nu(t_i) \tag{3.15}$$

with $0\leq t_1,<t_2<\ldots<t_n\leq T$.

Proof: Define the inverse of the mean value function and the stochastic process $\{K(X), x \geq 0\}$ as in the preceding paragraph. It follows that the quantities $M(t_i) \equiv X_i$ for $i = 1,2,\ldots,N$ are the first N waiting times from a Poisson process with intensity identically equal to unity and occurring in the interval $(0, M(T)]$. By theorem 1 the joint conditional density of $X_1,\ldots,X_N$ given that $N[M(T)] = n$ is

$$f(X_1, X_2, \ldots, X_n \mid N(M(T))=n) = \frac{n!}{[M(T)]^n} \qquad 0 \leq X_1 < X_2 < \ldots < X_n \leq M(T) \tag{3.16}$$

The absolute value of the Jacobian of the transformation which maps $(t_1, t_2, \ldots, t_n)$ into $(X_1, X_2, \ldots, X_n)$ is given by

$\left| \prod_{i=1}^{n} \frac{\partial X_i}{\partial t_i} \right|$, which is

$$|J| = \prod_{i=1}^{n} \nu(t_i) \quad . \tag{3.17}$$

It follows that the conditional density $g(t_1, t_2, \ldots, t_n \mid N(t)=n)$ is as given in the theorem. Note that the distribution function corresponding to the density $h(X)$ is given by

$$H(X) = \frac{M(X)}{M(T)} \qquad 0 \leq X \leq T \quad . \tag{3.18}$$

Theorem 4. Let $\{N(t), t \geq 0\}$ be a Poisson process with continuous mean value function $M(t)$. Under the condition that the waiting time to the nth event, t_n, is equal to T, the n-1 waiting times $t_1 < t_2 < \ldots < t_{n-1}$ in the interval $(0,T)$ at which events occur are random variables having the same distribution as the order statistics of a random sample of size n-1 from the probability density

$$h(X) = \frac{\nu(X)}{M(T)} \qquad 0 \leq X < T \tag{3.19}$$

where $\nu(X) = \frac{dM(x)}{dx}$ is the intensity function of the Poisson process.

That is,

$$g(t_1,t_2,\ldots,t_{n-1} \mid t_n=T) = \frac{(n-1)!}{[M(T)]^{n-1}} \prod_{i=1}^{n-1} \nu(t_i) \tag{3.20}$$

with $0 \le t_1 < t_2 < \ldots < t_{n-1} < T$.

Proof: The proof is the same as that of theorem 3 except that theorem 2 is applied instead of theorem 1 and the Jacobian is given by

$$|J| = \prod_{i=1}^{n-1} \nu(t_i) \quad . \tag{3.21}$$

Theorems 3 and 4 provide the basis for testing hypotheses concerning the mean value function of a Poisson process. Define the parameter m as follows

$$m = \begin{cases} n & \text{for the condition } N(T)=n \\ n-1 & \text{for the condition } t_n=T \end{cases} \tag{3.22}$$

in which n is the number of events observed in the interval $(0,T]$ of a stochastic process $\{N(t),t\ge 0\}$. Consider the hypothesis H_0:$N(t)$ is a Poisson process with continuous mean value function $M(t)$. If H_0 is true the waiting times $t_1,t_2,\ldots,t_m$ have the same distribution as the order statistics of a random sample of size m from the distribution $H(X)=M(X)/M(T)$. Define $H_m(X)$ to be $N(X)/N(T)$. Then the statistic

$$W_m^2 = m \int_0^T [H_m(X)-H(X)]^2 dH(X) \tag{3.23}$$

has the same distribution as the Cramér-von Mises statistic for a sample of size m from H(X). The statistic can be written in the form

$$W_m^2 = \frac{1}{12m} + \sum_{j=1}^{m} \left[\frac{M(t_j)}{M(T)} - \frac{2j-1}{2m}\right]^2 , \tag{3.24}$$

which is more suitable for computation.

If the mean value function contains an unknown parameter θ, then it is desirable to estimate the parameter from the data by calculating a statistic $\hat{\theta}_m$. If the estimator $\hat{\theta}_m$ satisfies the properties listed by Darling, then the statistic

$$C_m^2 = \frac{1}{12m} + \sum_{j=1}^{m} \left[\frac{M(t_j;\hat{\theta}_m)}{M(T,\hat{\theta}_m)} - \frac{2_j-1}{2m} \right]^2 \qquad (3.25)$$

may be used to test the hypothesis H_1: $\{N(t)t \geq 0\}$ is a Poisson process with mean value function $M(T;\theta)$ for some θ. The test is truly usable if the statistic C_m^2 is parameter-free.

4. THE RELIABILITY GROWTH PROCESS

Crow (8) has shown that the improvement in reliability of a complex system undergoing development in a test-fix-test-fix environment can be modeled by a certain family of nonhomogeneous Poisson processes. Crow (1) uses this same family of processes to represent the occurrences of failures in complex repairable systems. For this class of processes the mean value function is of the form

$$M(t) = \lambda t^{\beta} \qquad \lambda>0;\beta>0;t\geq 0 \qquad (4.1)$$

in which $\lambda>0$ can be interpreted as a scale parameter and $\beta>0$ as a shape parameter. The corresponding intensity function is

$$\nu(t) = \lambda\beta t^{\beta-1} \qquad t\geq 0 \quad . \qquad (4.2)$$

This family includes the homogeneous Poisson processes as the special case in which β equals unity.

The results of the preceding section can be used to derive a goodness of fit test for this class of processes. With the index m defined as in Section 3, it follows from theorems 3 and 4 that with

application of the appropriate condition the random variables $t_1, t_2, \ldots, t_m$ have the same distribution as the order statistics of a random sample for the cumulative distribution function

$$H(X) = \left(\frac{X}{T}\right)^{\beta} \qquad 0 \leq X \leq T \quad . \qquad (4.3)$$

With the appropriate choice of a estimator $\bar{\beta}$ the statistic

$$C_m^2 = \frac{1}{12m} + \sum_{j=1}^{m} \left[\left(\frac{t_j}{T}\right)^{\bar{\beta}} - \frac{2j-1}{2m} \right]^2 \qquad (4.4)$$

can be used to test the hypothesis that the observations $\{t_j\}$ are from Poisson process with mean value function of the form $M(t) = \lambda t^{\beta}$. Darling has shown that the distribution of this test statistic is independent of the true value of the parameter β. In fact it is distribution free over the class of distributions such that

$$F(X;\beta) = \left[R(X) \right]^{\beta} \qquad \beta > 0 \qquad (4.5)$$

for some cumulative distribution function $R(X)$.

As Crow has shown conditional maximum likelihood estimates of β can be derived from equation (3.15) for the case $N(T) = n$ and from equation (3.20) for the case $t_n = T$. These conditional maximum likelihood estimates are given by

$$\tilde{\beta} = \frac{m}{m \ln T - \sum_{i=1}^{m} \ln t_i} \qquad (4.6)$$

in which $m = n$ for conditioning on $N(T) = n$ and $m = n-1$ for conditioning

on $t_n = T$. This is a biased estimator for β with expected value

$$E(\tilde{\beta}) = \frac{m}{m-1}\beta \quad . \tag{4.7}$$

An unbiased estimate can thus be provided by

$$\bar{\beta} = \frac{m-1}{m \ln T - \sum_{i=1}^{m} \ln t_i} \quad . \tag{4.8}$$

It can be shown that the estimator $\bar{\beta}$ given in equation (4.8) should be used in equation (4.4) to calculate the statistic C_m^2. This must be done in order to satisfy the conditions needed for C_m^2 to have the limiting distribution described by Darling. In particular, the condition

$$\lim_{m\to\infty} mE(\bar{\beta}-\beta) = 0 \tag{4.9}$$

is satisfied since $\bar{\beta}$ is unbiased. This condition is not met by the estimator in equation (4.6).

5. DISTRIBUTION OF THE STATISTIC C_m^2

In order to use the statistic C_m^2 to test the goodness of fit hypothesis it is necessary to establish a table of critical values for selected significance levels. The small sample distribution of the statistic C_m^2 given in equation (4.4) is not analytically tractable. Moreover, the limiting distribution has only been defined in terms of its characteristic function. The distribution of C_m^2 has been determined through Monte Carlo simulation for values of m from 2 to 20 and for m equal to 30, 60, and 100.

The Monte Carlo simulation consists of repeated samples of

size m from the uniform distribution on the interval (0,1) and computation of

$$\overline{\beta} = \frac{m-1}{-\sum_{j=1}^{m} \ln U_j} \tag{5.1}$$

and

$$C_m^2 = \frac{1}{12m} + \sum_{j=1}^{m} \left[\left(u_j' \right)^{\overline{\beta}} - \frac{2j-1}{2m} \right]^2 \tag{5.2}$$

in which $\{u_j\}$ is the random sample and $\{u_j'\}$ is the corresponding set of order statistics. For each value of the index m there are 150,000 replications of this sampling.

Selected percentage points of the distribution of C_m^2 are presented in Table 1. The 1-α percentile of this distribution is to be used for a goodness of fit test with level of significance α. The accuracy of these percentage points can be determined by using the fact that any percentile of a random sample is asymptotically normal. Each sample of 150,000 actually consists of ten independent samples of size 15,000. The sample variance of estimate $\hat{C}_p$. of the p-th percentile is used to estimate the precision of the p-th percentile of the combined sample. Table 2 contains interval estimates of the percentiles of the distribution of C_m^2 with a confidence coefficient of .90.

The sample moments from the simulation can be used to determine how rapidly the distribution of C_m^2 is converging to the limiting distribution. Darling provided a means for calculating the moments of the limiting distribution. The mean, variance, and third central moment, μ_3, of the sampling distribution of C_m^2 and of the limit distribution are given in Table 3. The sample moments indicate the distribution for m equal to 100 matches the limiting distribution quite closely. The mean, variance, and third central moment from the simulation for m equal to 100 are each within one percent of the respective true moment of the limiting distribution. Hence the percentiles appearing in Table 1 for m equal to 100 can be used for larger values of m. Figure 1 contains plots of the empirically obtained density function for m equal to 5 and m equal to 100.

6. CONCLUSION

The percentiles in Table 1 provide a set of critical values for the Cramér-von Mises goodness of fit statistic for the case in which an exponential parameter is estimated. This table can be used to test the hypothesis that a random sample of size m comes from a parametric family in the class of distribution of the form

$$F(X;\theta) = \left(R(X)\right)^{\theta} \qquad \theta>0 \tag{6.1}$$

in which R(X) is some cumulative distribution function. The parameter θ is to be estimated from the data by an appropriate statistic.

The distribution of the Cramér-von Mises statistic can also be used to test hypotheses on the goodness of fit for certain stochastic processes. In particular, the hypothesis that a stochastic process is a member of the family of nonhomogeneous Poisson processes with mean value function of the form

$$M(t) = \lambda t^{\beta} \tag{6.2}$$

can be tested through use of the statistic.

TABLE 1

PERCENTILES FOR THE DISTRIBUTION OF C_m^2

M \ P	.80	.85	.90	.95	.99
2	.138	.149	.162	.175	.186
3	.121	.135	.154	.184	.23
4	.121	.134	.155	.191	.28
5	.121	.137	.160	.199	.30
6	.123	.139	.162	.204	.31
7	.124	.140	.165	.208	.32
8	.124	.141	.165	.210	.32
9	.125	.142	.167	.212	.32
10	.125	.142	.167	.212	.32
11	.126	.143	.169	.214	.32
12	.126	.144	.169	.214	.32
13	.126	.144	.169	.214	.33
14	.126	.144	.169	.214	.33
15	.126	.144	.169	.215	.33
16	.127	.145	.171	.216	.33
17	.127	.145	.171	.217	.33
18	.127	.146	.171	.217	.33
19	.127	.146	.171	.217	.33
20	.128	.146	.172	.217	.33
30	.128	.146	.172	.218	.33
60	.128	.147	.173	.220	.33
100	.129	.147	.173	.220	.34

TABLE 2

INTERVAL ESTIMATES OF PERCENTILES OF THE DISTRIBUTION C_n^2
90% CONFIDENCE COEFFICIENT

M \ P	.80	.85	.90	.95	.99
2	.1372-.1378	.1485-.1493	.1613-.1618	.1748-.1753	.1863-.1866
3	.1203-.1212	.1346-.1356	.1541-.1548	.1828-.1843	.2293-.2320
4	.1204-.1211	.1337-.1346	.1539-.1552	.1901-.1924	.2761-.2819
5	.1202-.1212	.1359-.1372	.1588-.1605	.1984-.2006	.2923-.2978
6	.1220-.1232	.1384-.1395	.1609-.1633	.2018-.2054	.3029-.3115
7	.1233-.1242	.1396-.1406	.1639-.1652	.2062-.2096	.3128-.3202
8	.1233-.1245	.1400-.1414	.1644-.1663	.2084-.2106	.3143-.3238
9	.1245-.1253	.1419-.1428	.1667-.1683	.2109-.2130	.3191-.3261
10	.1243-.1257	.1416-.1429	.1663-.1684	.2108-.2136	.3208-.3264
11	.1253-.1263	.1429-.1439	.1681-.1691	.2128-.2160	.3249-.3302
12	.1260-.1268	.1431-.1441	.1679-.1697	.2124-.2151	.3215-.3276
13	.1263-.1272	.1440-.1451	.1695-.1711	.2140-.2164	.3230-.3312
14	.1259-.1269	.1435-.1445	.1686-.1697	.2130-.2154	.3261-.3322
15	.1256-.1270	.1433-.1451	.1680-.1707	.2138-.2164	.3257-.3301
16	.1266-.1278	.1440-.1454	.1701-.1714	.2146-.2174	.3216-.3291
17	.1268-.1281	.1446-.1455	.1697-.1713	.2152-.2181	.3288-.3351
18	.1271-.1278	.1450-.1462	.1702-.1721	.2151-.2184	.3258-.3344
19	.1266-.1281	.1443-.1462	.1699-.1721	.2154-.2184	.3292-.3359
20	.1271-.1280	.1451-.1461	.1706-.1725	.2162-.2188	.3296-.3358
30	.1275-.1286	.1451-.1470	.1715-.1733	.2172-.2197	.3287-.3355
60	.1276-.1290	.1459-.1475	.1727-.1742	.2200-.2219	.3310-.3357
100	.1284-.1297	.1462-.1479	.1720-.1742	.2182-.2212	.3332-.3392

TABLE 3

MOMENTS OF THE DISTRIBUTION OF C_n^2

M	Mean	Variance	μ_3
2	.1124	.00082	.000026
3	.0929	.00179	.000107
4	.0898	.00267	.000281
5	.0892	.00319	.000409
6	.0894	.00347	.000471
7	.0899	.00375	.000550
8	.0897	.00383	.000565
9	.0903	.00393	.000550
10	.0902	.00391	.000546
11	.0906	.00405	.000599
12	.0906	.00405	.000602
13	.0910	.00410	.000609
14	.0908	.00409	.000603
15	.0909	.00411	.000619
16	.0911	.00411	.000599
17	.0914	.00418	.000643
18	.0915	.00421	.000639
19	.0914	.00418	.000617
20	.0914	.00424	.000625
30	.0917	.00425	.000632
60	.0920	.00429	.000629
100	.0922	.00432	.000644
∞	.0926	.00436	.000640

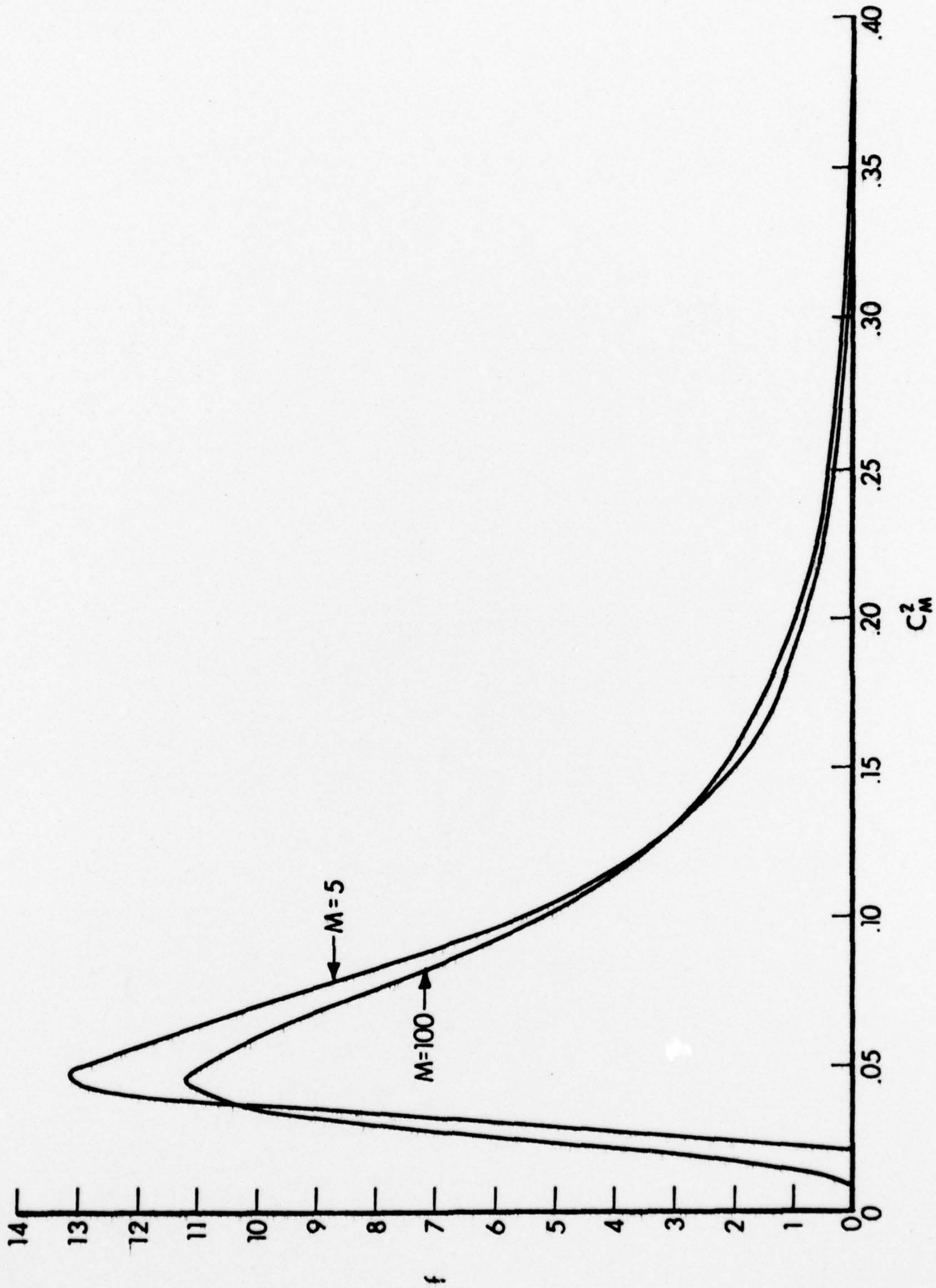

Figure 1. Density Function of C_M^2.

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